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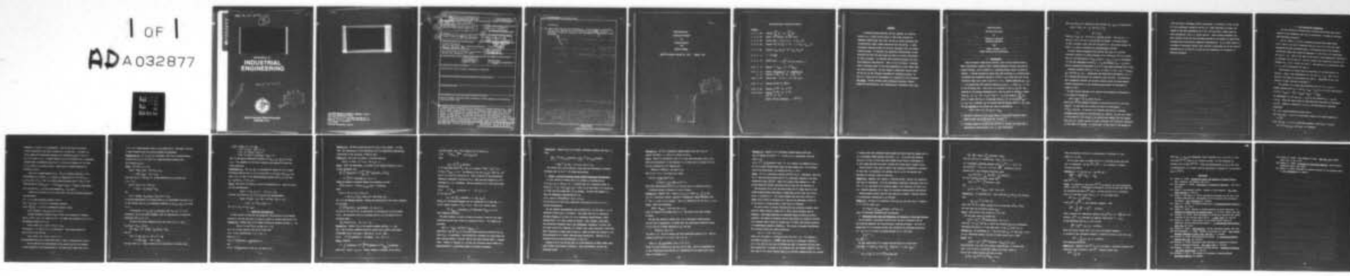

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A clearing process describes the net quantity in a service system (e.g. a batch service queue or dam) which receives an exogenous random input over time, and has an output mechanism that intermittently clears random quantities from the system. A semi-stationary clearing process is strictly stationary over its random clearing epochs. We describe the asymptotic distributions of such processes and show how they arise in limits of certain functionals of these processes. We identify some clearing processes with uniform asymptotic distributions. This is true for modulo c clearing with a stationary input if the Palm probability is →

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20 Abstract

used rather than the usual probability. We also present a functional central limit law and law of the iterated logorithm for clearing processes, as well as a result on the convergence of a sequence of such processes.

"Semi-Stationary

Clearing Processes"

by

Richard Serfozo

and

Shaler Stidham

NCSU-IE Technical Report No. 76-9 August, 1976

"Semi-Stationary Clearing Processes"

Errata:

p. 1, l. 9b: replace " $\sum_{k=0}^{n-1} Q'_n$ " by " $\sum_{k=0}^{n-1} Q'_k$ "

p. 2, l. 1b: insert "it" between "is" and "true"

p. 4, l. 14: should read: "$X\,(t_1 + u), \ldots, X\,(t_k + u)$"

p. 6, l. 10: Replace "B_+" by "$\mathcal{B}_+$"; l. 11: "$\ldots \xi\,\{A_k + T_{n_k} + h\}$"

p. 7, l. 6b: Replace "$X_{n-1}\,(W_{n-1})$" by "$X_{n-1}\,(W_{n-1}-)$"

p. 8, l. 1: "$- \hat{Y}\,[0, T^*_n)$"

p. 9, l. 9: should read: "$+ E \int_o^{W_o} f\,(X\,(u))\,du / E\,W_o \ldots$"

p.10, l. 6: Replace "$t^{-1} T_{n(t)}$" by "$t^{-1} T_{N(t)}$"

p.11, l. 1: Replace "Theorem 3.4" by "Theorem 3.5"

l. 4: Replace "$p\,(W_o > u)$" by "$P\,(W_o > u)$"

p.12, l. 7b: should read: "$Y\,(A + t, \omega) = Y(A, \theta_t \omega)$"

p.13, l. 2: Replace "$EY(1)$" by "$\hat{E}\hat{Y}(1)$"

p.16, l. 1b: Replace "$\hat{Y}^{(n)}$" by "$Y^{(n)}$"

p.17, l.2-3: Replace "$\hat{Y}^{(u)}$" by "$Y^{(n)}$" and "$\hat{Y}$" by "Y".

Delete "By the continuity ... $Y^{(n)} \xrightarrow{D} Y$."

ABSTRACT

A clearing process describes the net quantity in a service system (e.g. a batch service queue or dam) which receives an exogenous random input over time, and has an output mechanism that intermittently clears random quantities from the system. A semi-stationary clearing process is strictly stationary over its random clearing epochs. We describe the asymptotic distributions of such processes and show how they arise in limits of certain functionals of these processes. We identify some clearing processes with uniform asymptotic distributions. This is true for modulo c clearing with a stationary input if the Palm probability is used rather than the usual probability. We also present a functional central limit law and law of the iterated logorithm for clearing processes, as well as a result on the convergence of a sequence of such processes.
Key words: Service systems, semi-stationary processes, uniform asymptotic distributions, Palm probabilities, functional limit laws.

"Semi-Stationary

Clearing Processes"

by

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1. Introduction

Many stochastic input-output systems, such as batch service queues, dams, inventories, computer files, demand-responsive systems, and quality control systems, can be modeled as a clearing process, which we define as follows. A system receives an input over time according to a nondecreasing continuous-time stochastic process $Y = \{Y(t): t \geq o\}$ such that $Y(t)$ is the cumulative input up to time t and $Y(t) \to \infty$, a.s. Random quantities Q_o', Q_1', ... are cleared from the system at epochs $T_1 \leq T_2 \leq \dots$, where $T_n \to \infty$ a.s., by the following rule. After the n-th clearing at time T_n $(T_o = 0)$, the quantity in the system accumulates for a time A_n until it reaches a random level Q_n, i.e. $A_n = \inf\ \{u \geq 0: Y(T_n + u) - S_n \geq Q_n\}$ where $S_n = \sum_{k=0}^{n-1} Q_n'$. After a random (service or processing) time B_n, i.e. at the epoch $T_{n+1} = T_n + A_n + B_n$, a quantity Q_n' is cleared from the system, where $0 < Q_n' \leq Q_n$. The net quantity in the system over time is described by

$$X(t) = Y(t) - S_n, \text{ for } T_n \leq t < T_{n+1}.$$

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The n-th cycle of X, during the time interval $[T_n, T_{n+1})$ is described by

$$X_n(u) = Y(T_n + u) - S_n, \text{ for } 0 \leq u < W_n$$

$$= 0, \qquad \text{for } u \geq W_n$$

Where $W_n = T_{n+1} - T_n$. We call X a clearing process. When $Q_n = Q_n' = c$ and $B_n = 0$ a.s. for all $n \geq 0$, we call X a modulo c clearing process. In such a process, every time the net quantity in the system reaches the level c, c units of quantity are instantaneously removed.

We shall assume throughout this article that $\{X_n\}$ is a strictly stationary sequence of random elements (X is stationary over its cycles). That is, the distribution of $X_{n_1+h}(t_1),\ldots, X_{n_k+h}(t_k)$ is independent of h for any nonnegative reals $t_1,\ldots, t_k$ and integers $n_1,\ldots, n_k$ and h. This is equivalent (Proposition 2.1) to X being semi-stationary over T [11].

Clearing processes with $Q_n = c$, $B_n = 0$, $X(T_n) = 0$ (i.e., $Q_n' = Y(T_{n+1}) - S_n$), and with $X_o, X_1,\ldots$ independent and identically distributed (i.e., X is a regenerative process) are studied in [13] and [14]. In particular, their asymptotic behavior and optimal clearing levels are investigated. Possible applications of the clearing-process model are discussed at length in [14].

In this article (Section 3) we describe the asymptotic distribution p of X, which is defined by

$$p(B) = \lim_{t\to\infty} t^{-1}\int_o^t I_B(X(u))\,du \text{ a.s.}$$

where $I_B(.)$ is the indicator function of the Borel set B in R, the real numbers. We then show how p arises in limits of functionals $t^{-1}\int_o^t f_t(X(u))du$. In Section 4 we identify some modulo c clearing processes whose asymptotic distributions are uniform. Our aim here (which is what motivated this study) is to provide some insight into the anomaly that only for a small subclass of input processes with stationary increments is true that p is uniform. In particular, we show that if the inverse of

Y has stationary increments, then p is uniform. A corollary to this is that if Y has stationary increments and the a.s. limit defining p is taken with respect to the Palm probability of Y ([3], [5] and [10]), rather than the usual probability, then p is indeed uniform. These results supplement the basic results in [8] and the references therein on the subject of identifying processes that have uniform asymptotic distributions. In our final Section 5, we present a functional central limit law and a functional law of the iterated logarithm for X, as well as a result on the convergence of a sequence of clearing processes.

2. The Stationarity Assumption

In this section we introduce more notation and discuss some consequences of our assumption that $\{X_n\}$ is stationary. We begin by defining semi-stationary processes.

Let $X = \{X(t): t \geq 0\}$ be a stochastic process with sample paths in $D = D[0,\infty)$, the set of functions from $R_+ = [0,\infty)$ to R that are right continuous and have left-hand limits. We associate with D the smallest σ-field that makes the projection mappings $x \to x(t)$ from D to R measurable for each $t \in R_+$. This is the same as the Borel σ-field generated by Skorohod's J_1 topology, see [2] and [7]. We let θ_t denote the translation operator on D defined by $\theta_t\, x\,(u) = x(t+u)$ for each u. The operator θ_t is measurable since $x \to \theta_t\, x\,(u)$ is measurable for each u.

The process X is called strictly stationary if the distribution of $X(t_1+u),\ldots(X(t_k + u)$ is independent of u for all $t_1,\ldots,t_k$. That is, if $\theta_u X$ and X are equal in distribution for each u. (Hereafter we use "stationary" to mean "strictly stationary.")

The process X is called semi-stationary over $T = \{T_n: n \geq 0\}$, a random sequence with $T_0 = 0$ and $W_n = T_{n+1} - T_n \geq 0$ a.s., if the distribution of $X(t_1 + T_{n_1+h}),\ldots,X(t_k + T_{n_k+h}),\ W_{n_1+h},\ldots,\ W_{n_k+h}$ is independent of h for any $t_1,\ldots,t_k$ in R_+ and integers $n_1,\ldots,n_k$ and h in R_+. These and other types of semi-stationary processes are discussed in [11]. Similar to Theorem 2.2 in [11], the following are equivalent statements:

(i) X is a semi-stationary process over T.

(ii) $\{(\theta_{T_n} X, W_n) : n \geq 0\}$ is a stationary sequence (of random elements of $D \times R$).

(iii) $\{(X_n, W_n) : n \geq 0\}$ is stationary sequence, where $X_n(t) = X(T_n+t)$ for t in $[T_n, T_{n+1})$, and $X_n(t) = 0$ elsewhere.

(iv) For any $f : D \to R$ the process $X'(t) = f(\theta_t X)$ $t \in R_+$, is semi-stationary over T.

We shall frequently use, without mention, the well-known fact that (i) is equivalent to (iv) for stationary processes. This by the way, is the key to proving the above equivalences. Note that a semi-stationary process X over T in which $\{(X_n, W_n) : n \geq 0\}$ are independent and identically distributed, is a regenerative process with regeneration times T_n.

For the remainder of this article we assume that X is a clearing process as described in Section 1 with clearing levels Q_n, clearing quantities Q_n' and clearing times T_n. Our assumption that the cycles $\{X_n\}$ are stationary has the following characterization.

<u>Proposition</u> 2.1. The sequence $\{X_n\}$ is stationary if and only if X is semi-stationary over T.

<u>Proof.</u> For each $n \geq 0$ we can write

$$W_n = 0 \quad \text{if } X_n(.) = 0$$

$$= \inf \{u > 0 : X_n(u) = 0\} \quad \text{otherwise.}$$

That is $W_n = f(X_n)$ where f is a measurable function from D to R_+. Consequently, if $\{X_n\}$ is stationary, then $\{(X_n, W_n)\}$ is stationary and so X is semi-stationary over T. The converse assertion is obvious.

Note that the stationarity of $\{X_n\}$ implies that $W_n = A_n + B_n$ $(n \geq 0)$ is a stationary sequence, and that $Y(0)$ and $X(T_n)$ have the same distribution for each n. We shall continue this discussion after we define semi-stationary random measures.

Let M denote the set of positive measures on R_+ that are finite on compact sets (i.e. Radon measures). For $\mu \in M$ we write μA or $\mu\{A\}$ as the μ-measure of A in B_+, the Borel sets of R_+, and we let $\mu(t) = \mu[0,t]$ for $t \in R_+$ denote the cumulative distribution function (c.d.f.) of μ. Note that $\mu(.) \in D$. We associate with M the smallest σ-field that makes the mappings $\mu \to \mu A$ for $A \in \mathcal{B}_+$ measurable. This is the same as the Borel σ-field generated by the vague topology on M, [1]and [6] . As above, we let θ_t denote the translation operator on M defined by $\theta_t \mu(A) = \mu(A + t)$ for $A \in \mathcal{B}_+$ and $t \in R_+$. A random measure ξ on R_+ is defined to be a measurable function from a probability space to M. If ξA is an integer a.s. for each $A \in \mathcal{B}_+$, then ξ is a point process. Some basics of random measures are discussed in [6] .

Let ξ be a random measure on R_+. The ξ is called stationary if the distribution of $\xi\{A_1+t\},\ldots, \xi\{A_k+t\}$ is independent of t for all $A_1,\ldots, A_k$ in B_+ and $t \in R_+$. The ξ is called semi-stationary over $T = \{T_n\}$ if the distribution of $\xi\{A_1+T_{n_1+h}\},\ldots, \xi\{A_k+T_{n_k+h}\}, W_{n_1+h},\ldots,W_{n_k+h}$ is independent of h for each $A_1,\ldots,A_k$ in $\mathcal{B}_+$ and integers $n_1,\ldots,n_k$ and h.

Similar to the above, one can easily show that the following statements are equivalent:

(i) ξ is a semi-stationary measure over T.

(ii) $\{ \theta_{T_n} \xi, W_n) : n \geq 0\}$ is a stationary sequence.

(iii) $\{(\xi_n, W_n) : n \geq 0\}$ is a stationary sequence, where

$$\xi_n A = \xi(\{A+T_n\} \cap [T_n, T_{n+1}))\} \text{ for } A \in \mathcal{B}_+.$$

Another important concept we use is that of an inverse of a measure. Let $M_\infty = \{\mu \in M : \mu R_+ = \infty\}$, which is in M. We define the inverse of $\mu \in M_\infty$ to be the measure $\hat{\mu}$ whose c.d.f. is

$$\hat{\mu}(t) = \inf\{s \geq 0 : \mu(s) > t\} \text{ for } t \in R_+.$$

The mapping $\mu \to \hat{\mu}$ from M_∞ to M_∞ is measurable. This follows since for each α and t in R_+ the set

$$\{\mu \in M_\infty : \hat{\mu}(t) < \alpha\} = \{\mu \in M_\infty : \mu(\alpha-) > t\}$$

is a measurable set in M_∞, which means that $\mu \to \hat{\mu}(t)$ is measurable for each t.

We now show how the stationarity of the cycles $\{X_n\}$ of X is related to the structure of the input process Y. To this end we shall view Y as a c.d.f. of a random measure which we also denote by Y. Note that $\hat{Y}$ is also a random measure since the inverse mapping is measurable.

Proposition 2.2. If $\{X_n, Q_n'\}$ is stationary, then Y is a semi-stationary measure over T_n $(n \geq 1)$, and $\hat{Y}$ is a semi-stationary measure over $T_n^* = Y[0,T_n)$ $(n \geq 1)$

Proof. For each $n \geq 0$, let

$$Y_n(t) = Y[T_n, T_n+t] \quad \text{for } 0 \leq t < W_n$$
$$= Y[T_n, T_{n+1}) \quad \text{for } t \geq W_n.$$

Note that $Y_n(.) = 0$ if $W_n = 0$. By the definition of X_n we have for $0 \leq t < W_n$,

$$Y_n(t) = X_n(t) + S_n - Y[0, T_n)$$
$$= X_n(t) - X_{n-1}(W_{n-1}) + Q'_{n-1}$$

and

$$Y_n(t) = X_n(W_n) - X_{n-1}(W_{n-1}) + Q'_{n-1} \text{ for } t \geq W_n.$$

In proving Proposition 2.1 we showed that W_n is a measurable function of X_n. It then follows that Y_n is a measurable function of $\{X_n, X_{n-1}, Q'_{n-1}\}$ for each $n \geq 1$.

Consequently, if $\{X_n, Q_n'\}$ is stationary then (Y_n, W_n) $(n \geq 1)$ is stationary, and by the above comments, this is equivalent to Y being semi-stationary over T_n $(n \geq 1)$.

To prove the second assertion first note that for $n \geq 1$, $T_n^* = X_{n-1}(W_{n-1}) + S_{n-1}$, and so

$$W_n^* = T_{n+1}^* - T_n^* = X_n(W_n) - X_{n-1}(W_{n-1}) + Q'_{n-1}$$

Now let

$$\hat{Y}(t) = \hat{Y}\,[T_n^*, T_n^* + t] \quad \text{for } 0 \leq t < W_n^*$$
$$= \hat{Y}\,[T_n^*, T_{n+1}^*) \quad \text{for } t \geq W_n^*.$$

For $W_n^* > 0$ and $0 \leq t < W_n^*$ it follows by the properties of inverses that

$$\hat{Y}_n(t) = \hat{Y}(T_n^* + t) - \hat{Y}[\,),T_n^*)$$
$$= \inf \{u: Y(u) > T_n^* + t\} - T_n$$
$$= \inf \{u: Y(T_n+u) > T_n^* + t\}$$
$$= \inf \{u: X_n(u) > X_{n-1}(W_{n-1}^-) - Q'_{n-1} + t\}$$

Thus, $\hat{Y}_n$ and W_n^* are measurable functions of (X_{n-1}, X_n, Q'_{n-1}) for each $n \geq 1$. The argument for the first assertion now applies to yield the second assertion.

Proposition 2.3. The $\{X_n, Q'_n\}$ is stationary if either (i) Y is a semi-stationary measure over T_n $(n \geq 0)$ with $Y(0) = 0$ and $Q'_n = Y(T_n, T_{n+1}]$ for each $n \geq 0$, or (ii) $\hat{Y}$ is a semi-stationary measure over S_n $(n \geq 0)$ and $Q_n = Q'_n$ and $B_n = 0$ for $n \geq 0$.

Proof. This proof is similar to that for Proposition 2.2. When (i) holds use the representation

$$X_n(t) = Y\,[T_n, T_n + t) \qquad \text{for } 0 \leq t < W_n.$$

And when (ii) holds use $T_n = \hat{Y}\,[0, S_n)$ and the representation

$$X_n(t) = Y(T_n+t) - S_n = \inf \{u: \hat{Y}(u) > T_n + t\} - S_n$$
$$= \inf \{v: \hat{Y}\,[S_n, S_n+v] > t\}$$

for $0 \leq t < W_n = \hat{Y}\,[S_n, S_{n+1})$.

3. Asymptotic Distributions

In this section we describe the asymptotic distribution of the clearing process X, and then show how it arises in limits of certain functionals of X.

Theorem 3.1. Suppose $\{X_n\}$ is an ergodic stationary sequence and $EW_0 < \infty$. Let

$$p[0,x] = E \min \{\hat{Y}(x), W_0\}/EW_0 \quad \text{for } x \in R_+.$$

Let f be a Borel function from R_+ to R such that

(3.1) $E \sup \{|\int_0^t f(Y(u)du|: 0 \leq t < W_0\} < \infty.$

Then as $t \to \infty$,

(3.2) $t^{-1}\int_0^t f(X(u)du \to \int_{R_+} f(x)dp(x)$ a.s.

and

(3.3) $t^{-1}\int_0^t I_B(X(u))du \to p(B)$ a.s. for each $B \in \mathcal{B}_+$.

Remarks 3.2. The above assertions hold if $\{X_n\}$ is not ergodic. In this case, the expectations in the definition of p are conditional expectations conditioned on the invariant σ-field of $\{X_n\}$.

Remark 3.3. Note that for modulo c clearing processes

$$p[0,x] = E\,\hat{Y}(x) \,/\, E\,\hat{Y}\,[0,c) \quad \text{for } 0 \le x < c.$$

Proof. Under the hypotheses, it follows by the ergodic theorem for stationary sequences that

$$T_n^{-1} \int_0^{T_n} f(X_n(u))du = n^{-1} \sum_{k=0}^{n-1} \int_{T_k}^{T_{k+1}} f(X_k(u))du \,/\, (n^{-1}T_n)$$

$$\to E \int_0^{W_0} f(Y(u))du/EW_0 \text{ a.s. as } n \to \infty.$$

Using the change of variable formula for integrals, we have

$$\int_0^{W_0} f(X(u))du = \int f(Y(u))\, I_{[0,W_0]}(u)du = \int f(x)d\xi(x),$$

where

$$\xi[0,x] = \lambda\{0 \le u \le W_0 : Y(u) \le x\}$$

$$= \sup\{0 \le u \le W_0 : Y(u) \le x\} = \min\{\hat{Y}(x), W_0\},$$

and λ is the Lebesque measure. Taking the expectation of the latter integral, we then have

$$T_n^{-1} \int_0^{T_n} f(X_n(u))du \to \int_{R+} f(x)dp(x) \quad \text{a.s. as } n \to \infty.$$

In view of this and (3.1), an application of Corollary 5.1 of [12] yields (3.2). The assertion (3.3) is a special case of (3.2) in which (3.1) is satisfied since

$$\int_0^t I_B\,(X(u))du \le W_0 \quad \text{for } 0 \le t < W_0.$$

Theorem 3.4. Suppose $\{X_n\}$ is an ergodic sequence and $EW_0 < \infty$. Let $f_t(t \in R_+)$ be Borel functions from R to R such that $|f_t(x)| \le g(x)$ where g satisfies (3.1), and $f_t(x) \to f(x)$ for p - a.e.x. Then

$$(3.4) \qquad t^{-1} \int_0^t f_t(X(u))du \to \int f(x)dp(x) \quad \text{a.s. as } t \to \infty.$$

Proof. Clearly

$$t^{-1} \int_0^t f_t(X(u))du = t^{-1} \int_0^{T_{N(t)}} f_t(X(u))du + t^{-1}\int_{T_{N(t)}}^t f_t(X(u))du,$$

where $N(t) = \sup(n : T_n \le t)$. Using a change of variable (as we did in the last proof), the first integral can be written as

$$(3.5) \qquad t^{-1}T_{N(t)} \int_0^{S_{N(t)}} f_t(x)\, \xi_{N(t)}(dx)$$

where

$$\xi_n B = T_n^{-1} \int_0^{T_n} I_B\,(X(u))du.$$

It is well-known that $n^{-1}T_n \to EW_o$ a.s. implies $t^{-1}N(t) \to EW_o^{-1}$ a.s., so that $t^{-1}T_{n(t)} \to 1$ a.s. By Theorem 3.1 we have $\xi_{N(t)}B \to p(B)$ a.s. for each $B \in \mathcal{B}_+$ and (3.2) holds for g. Thus it follows by the generalized dominated convergence theorem in [9, p. 230] that the integral in (3.5) converges a.s. to $\int f(x)dp(x)$. Now the assertion (3.4) will follow upon showing that

$$(3.6) \qquad t^{-1} \int_{T_{N(t)}}^t f_t(X(u))du \to 0 \qquad \text{a.s. as } t \to \infty.$$

To see this let

$$M_n = \sup\,\{\int_{T_n}^t g\,(X(u))du : t \in [T_n, T_{n+1})\}.$$

The M_n is a stationary sequence and our assumption (3.1) says $EM_1 < \infty$. By the ergodic theorem, $n^{-1}M_n \to 0$ a.s. Then (3.6) follows since

$$t^{-1}\left| \int_{T_{N(t)}}^t f_t(X(u))du\right| \leq (t^{-1}N(t))(N(t)^{-1}M_{N(t)}) \to 0 \text{ a.s.}$$

This completes the proof.

The above results, as well as those in Section 5, hold for any semi-stationary process that arises in conjunction with X. For example the processes

$$U(t) = t - T_{N(t)} \quad \text{and} \quad V(t) = T_{N(t)+1} - t$$

where $N(t) = \sup\,\{n: T_n \leq t\}$, are each semi-stationary processes over T_n, since $\{W_n\}$ is stationary. The U(t) and V(t) represent the times since the last clearing before t, and the time to the next clearing after t, respectively. Similar to Theorem 3.1, we have the following result, which is a generalization to a well-known result for renewal processes.

Theorem 3.4. Suppose $\{W_n\}$ is an ergodic stationary sequence with $EW_0 < \infty$. Then

$$\lim_{t\to\infty} t^{-1} \int_0^t I_{[0,x]}(U(s))ds = \lim_{t\to\infty} t^{-1} \int_0^t I_{[0,x]}(V(s))ds$$

$$= (EW_0)^{-1} \int_0^x p(W_0 > u)\, du \text{ a.s. for } x \in R_+.$$

The results here and in Section 5 also hold for functionals of several processes such as for $t^{-1} \int_0^t f(X(s),U(s),V(s))ds$.

4. Modulo c Clearing Processes Having Uniform Asymptotic Distributions

In this section we assume that X is a modulo c clearing process (i.e. $Q'_n = Q_n = c$ and $B_n = 0$). A special case is a clearing process in which $Q_n = c$, $Q'_n = Y(T_n, T_{n+1}]$, $B_n = 0$ (for $n \geq 0$) and either Y is continuous or Y has only unit jumps and c is an integer. We know from Remark 3.3 that the asymptotic distribution of X is

(4.1) $p[0,x] = E\hat{Y}(x)/E\hat{Y}(c-)$ for $x \in [0,c)$.

Our aim here is to give some sufficient conditions on Y for p to be a uniform distribution.

For the special deterministic case in which $Y(t) = at$ for some $a > 0$, it obviously follows that p is uniform. This might lead one to conjecture (without knowing (4.1)) that if Y is a stationary random measure, which implies $EY(t) = at$, then p would be uniform. This, however, even under the additional assumption that Y has independent increments, is false [13]. Our major result here (Theorem 4.3) asserts that, under some minor conditions, this conjecture is indeed true if the a.s. limit is taken with respect to the Palm probability of Y rather than the usual probability. Theorem 4.3 is obtained from the following Theorems 4.1 and 4.2.

Looking at(4.1) one can see that it is the linearity of $E\hat{Y}(x)$ rather than that of $EY(t)$ that yields a uniform p. More specifically, we have the following result.

Theorem 4.1. If $\hat{Y}$ is a stationary random measure such that $\{X_n\}$ is ergodic and $E\hat{Y}(1) < \infty$, then p is uniform on $[0,c)$.

Proof. Since $\hat{Y}$ is stationary, then it is also semi-stationary over $S_n = nc$. Then from Proposition 2.3 and Theorem 3.1, we know that p is given by (4.1). Now the stationarity of $\hat{Y}$ implies that

$$E\hat{Y}(0,x] = x\, E\hat{Y}(0,1] \quad \text{for each } x \geq 0.$$

Also $\hat{Y}\{x\} = 0$ a.s. for each $x \geq 0$, since

$$E\hat{Y}\{x\} = E \lim_{n\to\infty} \hat{Y}(x-n^{-1},x]$$

$$= E\,\hat{Y}(0,x] - \lim_{n\to\infty} E\hat{Y}(0,x-n^{-1}] = 0.$$

From these observations and (4.1) it follows that p is uniform on $[0,c)$.

A discrete analog of Theorem 4.1 is as follows.

Theorem 4.2. Suppose $Y(t) = \sup\{n \geq 0: Z_n \leq t\}$, where $Z_n = \sum_{k=1}^{n} Y_k$ and $\{Y_k\}$ is a stationary ergodic sequence of nonnegative random variables with $0 < EY_1 < \infty$, and c is an integer. Then p is uniform on the set $\{0,1,\ldots,c-1\}$.

Proof. Under the hypotheses

$$E\hat{Y}(x) = EZ_{[x+1]} = [x+1]\, EY_1 \quad \text{for } x \geq 0,$$

where $[u]$ denotes the integer part of u. The rest of the proof follows as above.

For the next result we assume that Y is a stationary random measure. With no loss in generality, we assume that the underlying probability space Ω is M, the set of Radon measures on R_+, and that

$$Y(A+t,\omega) = Y(A\;\theta\;\omega)$$

for each $\omega \in \Omega$. We let $\hat{P}$ denote the Palm probability measure of Y. This is defined on the set $\hat{\Omega} = \{\mu \in M: \hat{\mu}\;(0) = 0\}$ by

$$\hat{P}(A) = E \int_0^1 I_B(\theta_t Y)\,dY(t) \quad \text{for } A \in \hat{F} = \hat{\Omega} \cap F,$$

where F is the σ-field on Ω, see [3], [5] or [10]. The $\hat{P}$ is interpreted to be the "conditional distribution of Y conditioned on the event that Y has a point of increase at 0."

Theorem 4.3. Suppose Y is a stationary random measure such that $\{X_n\}$ is ergodic and $EY(1) < \infty$. If $Y(t)$ is a.s. continuous, then for each $B \in \mathcal{B}_+$

(4.2) $\quad t^{-1} \int_0^t I_B(X(u))du \to p\{B\} \quad \hat{P}$ - a.s., where p is uniform on $[0,c)$.

If Y is a point process whose atoms are unit masses and c is an integer, then (4.2) holds where p is uniform on $\{0,1,\ldots,c-1\}$.

Proof. If Y is a stationary measure and $Y(t)$ is a.s. continuous, then from [3, p. 218] and [15] we know that $\hat{Y}$ is a stationary measure on the Palm probability space $(\hat{\Omega},\hat{F},\hat{P})$. Then (4.2) follows from Theorem 4.1. The second assertion follows similarly from [3] and [15] and Theorem 4.2.

The assertion (4.2) with the Palm probability $\hat{P}$, means that X is uniformly asymptotically distributed, conditioned on the event that Y has a point of increase at 0. The latter insures that the first cycle of X begins with a point of increase of Y, just as the subsequent cycles do by the nature of clearing when a given level is attained.

The above result is nicely illustrated by the classical economic order quantity (EOQ) inventory model [16, p. 803], which is widely used in industry. This model describes an inventory system in which an initial quantity c is placed in inventory and as time passes the inventory declines at a constant rate a until it reaches zero, at which time another quantity c is immediately placed in inventory. This cycle is repeated indefinitely. The inventory level over time is given by

$$Z(t) = c - X(t) \quad \text{for } t \geq 0,$$

where X is our modulo c clearing process with $Y(t) = at$. The quantity c (the EOQ) is given by $c = \sqrt{2aK/h}$, when the cost of ordering a quantity x is $K + bx$, and h is the cost of holding one unit of inventory per unit time. This formula for c is based on the property that the average inventory level over time is $c/2$, which follows since X is uniformly asymptotically distributed.

It appears that this inventory model should also apply when the demand (our Y) is a stationary random measure with $EY(t) = at$. We can now say (knowing Theorem 4.3) that it does apply when demand occurs either continuously or in unit jumps and when one starts viewing the process when a demand occurs.

We now return to the case in which Y has stationary independent increments. We know that p is generally not uniform, but it is for the special case described as follows, which is from [8].

<u>Remark 4.4.</u> Suppose X is a modulo 1 clearing process, without the assumption that $\{X_n\}$ is stationary. If Y has stationary independent increments such that $Y(0) = 0$ a.s. and there is no rational number r for which all the $Y(t)$'s are a.s. integral multiples of r, then X is uniformly asymptotically distributed.

Another general result from [8] that applies to modulo c clearing (without our stationarity assumption) is as follows:

<u>Remark 4.5.</u> If there is a function ψ from R_+ to R_+ such that $\psi(x) \to 0$ implies $x \to 0$, and for some $a \geq 0$,

$$\int_a^\infty t^{-1} E\psi(t^{-1} \int_0^t I_{[0,x)}(X(u))du - xc^{-1})dt < \infty,$$

then X is uniformly asymptotically distributed.

5. <u>Functional Limit Laws and Convergence of Sequences of Clearing Processes</u>

We now present a functional central limit law and a functional law of the iterated logarithm for semi-stationary clearing processes. For this we assume that X is a clearing process that satisfies the following conditions:

(i) $\{X_n\}$ is a stationary ϕ_n-mixing sequence [2, p. 166] with

$$\sum_n \phi_n^{1/2} < \infty.$$

(ii) $EW_0^2 < \infty$.

We also assume that f is a Borel function from R to R such that:

(iii) $EM_1^2 < \infty$, where $M_n = \sup\{|\int_{T_n}^t f(x(u))du : t \in [T_n, T_{n+1})\}$.

Let $a = EW_0$, $A = a^{-1} E \int_0^{W_0} f(x(u))du$ and

$$(Y_n^1, Y_n^2) = (W_n - a, \int_{T_{n-1}}^{T_n} f(X(u))du - AW_n).$$

From (i) and (ii) it follows that: $\{(Y_n^1, Y_n^2): n \geq 0\}$

is a stationary ϕ_n-mixing sequence with $\sum_{n=1}^{\infty} \phi_n^{1/2} < \infty$, $EY_1^1 = 0$,

and $E(Y_1^i Y_2^j) < \infty$ for $i,j = 1,2$ [2,p. 166]. Furthermore, by [2, Theorem 20.1], the following quantities exist

$$\sigma_{ij} = E(Y_1^i Y_2^j) + \sum_{k=2}^{\infty} E(Y_1^i Y_k^j) + \sum_{k=2}^{\infty} E(Y_k^i Y_1^j).$$

For the following result we assume that

$$Z_n(t) = n^{-1/2} \left(\int_0^{nt} f(X(u))du - nAt \right)$$

and

$$Z_n'(t) = n^{-1/2} (T_{[nt]} - nat) \quad \text{for } t \geq 0.$$

<u>Theorem 5.1.</u> If conditions (i) - (iii) hold and σ_{11} and σ_{22} are positive, then

$$(Z_n, Z_n') \xrightarrow{\mathcal{D}} (a^{-1/2} W^2, W^1)$$

where (W^1, W^2) is a Weiner process in $D \times D$ such that $(W^1(1), W^2(1))$ has zero mean and covariance matrix $\{\sigma_{ij}\}$.

<u>Proof.</u> This follows from Corollary 5.2 in [12].

For the next result we assume that

$$Z_n^*(t) = B_n^{-1} \left(\int_0^{nt} f(x(u))du - nAt \right) \quad \text{for } t \geq 0,$$

where $B_n^2 = 2n\, \sigma_{22}^2 \log \log n$, and we let K denote the set of absolutely continuous functions $x \in D$ such that $x(0) = 0$ and $\int_0^{\infty} x'(t)^2\, dt < 1$.

<u>Theorem 5.2.</u> If conditions (i) - (iii) hold and $\sigma_{22} > 0$, then $\{Z_n^*: n \geq 3\}$ is a.s. relatively compact with set of limit points $\{x(a\cdot): x \in K\}$.

<u>Proof.</u> Let

$$Z_n^{\#}(t) = B_n^{-1} \left(\int_0^{T_{[nt]}} f(x(u))du - AT_{[nt]} \right) \quad \text{for } t \geq 0.$$

From Corollary 3 in [12], with obvious modifications, it follows that $\{Z_n^{\#} : n \geq 3\}$ is a.s. relatively compact with limit points K.

Also by the ergodic theorem and (iii) we have

$$B_n^{-1} M_n = (B_n^{-1} n^{1/2}) (n^{-1/2} M_n) \to 0 \text{ a.s.}$$

Thus the assertion follows by an application of Corollary 2.3 and Remark 2.4 in [12].

For our next result we assume that X is a clearing process such that $\{X_n\}$ is stationary. We let $X^{(1)}, X^{(2)}, \ldots$ be a sequence of similar clearing processes.

Theorem 5.3. If for any $m_1, \ldots, m_k$ and $k \geq 1$,

$$(X_{m_1}^{(n)}, \ldots, X_{m_k}^{(n)}) \xrightarrow{\mathcal{D}} (X_{m_1}, \ldots, X_{m_k})$$

then $X^{(n)} \xrightarrow{\mathcal{D}} X$.

Proof. It suffices to show that $X^{(n)} \xrightarrow{\mathcal{D}} X$ on $D[0,s]$, for any X-continuity point s, see [7]. Furthermore by Theorem 5.1 of [2] we need only show this for nonrandom functions $X^{(n)}$ and X. To this end let s be a continuity point of X and pick m such that $T_m > s$. We can always write

$$X^{(n)}(t) = \sum_{k=0}^{\infty} X_k^{(n)}(t - T_k)$$

since $X_1^{(n)}, X_2^{(n)}, \ldots$, have disjoint supports. Then

$$X^{(n)} = \sum_{k=0}^{m} X_k^{(n)} \circ \phi_k^{(n)} \quad \text{on } [0,s],$$

where o denotes the composition operator and $\phi_k^{(n)}(t) = t - T_{k-1}^{(n)}$ for $t \geq 0$. Clearly, $\phi_k^{(n)} \to \phi_k$ uniformly on compacts, where $\phi_k(t) = t - T_{k-1}$. By the hypothesis and [2, p. 143] we have

$$X_k^{(n)} \circ \phi_k^{(n)} \to X_k \circ \phi_k \quad \text{on } D[0,s].$$

Since the latter limits (for $0 \leq k \leq m$) have disjoint supports, it follows by the continuity property of addition in $D[0,s]$, see [17], that

$$X^{(n)} \to \sum_{k=1}^{m} X_k \circ \phi_k = X \quad \text{on } D[0,s].$$

This completes the proof.

Example 5.4. Suppose X and $X^{(n)}$ $(n \geq 1)$ are modulo c clearing processes with respective input processes Y and $Y^{(n)}$. Suppose further that

$$\hat{Y}^{(n)} = \sum_{k=1}^{n} \xi_{nk}$$

where $\xi_{n1},\ldots,\xi_{nn}$ are independent random measures on R_+, as in [6, p. 232], such that $\hat{Y}^{(n)} \overset{\mathcal{D}}{\to} \hat{Y}$ where $\hat{Y}$ is a Poisson process. By the continuity of inverses [17] we have $Y^{(n)} \overset{\mathcal{D}}{\to} Y$. Using the representation of X_n in the proof of Proposition 2.3 it follows that the hypothesis of Theorem 5.3 is satisfied, and so $X^{(n)} \overset{\mathcal{D}}{\to} X$

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